# Numerical Solutions for Laminar Incompressible Flow past a Paraboloid of Revolution

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**Numerical solutions are obtained to the Navier-Stokes equations for flow past a paraboloid of revolution at zero angle of attack for a full range of Reynolds numbers. The solution approaches the Stokes and Oseen results at low Reynolds number and approaches the boundary-layer solution at high Reynolds number. The numerical method employed is an implicit alternating direction method which converges to the steady-state solution rapidly. Numerical experiments are made to determine the optimal time step for maximum rate of convergence.**

## Introduction

THERE have been several interesting developments in the last few years in the approximate solution of the Navier-Stokes equations which should be of use in developing fast and accurate techniques for finding exact numerical solutions to the full set of equations.

The first development was that of Blottner and Flügge-Lotz,[1] Smith and Clutter,[2] and others who developed numerical techniques for solving the boundary-layer equations. Later refinements have made their methods more accurate and faster, making them practical techniques for solving even boundary-layer problems involving chemical reactions. Two important features of Blottner's original and later work can be used to our advantage. First, the use of similarity type variables can be exploited. The use of similarity variables (see Blottner[3]) in boundary-layer calculations reduces numerical inaccuracies since variation of the solution with respect to one coordinate direction may be small and therefore larger step sizes can be taken in that direction than could be taken in nonsimilar variables. In addition, the governing equations in these variables automatically produce starting solutions. Second, Blottner found that implicit methods for solving the boundary-layer equations are in general far superior to explicit methods. The same should therefore be true for the Navier-Stokes equations, as far as the spatial derivatives are concerned, if problems with boundary-layer type behavior are being solved. A method for solving the Navier-Stokes equations which makes use of all of the knowledge and experience gained in solving the boundary-layer equations should therefore be advantageous. Any new improvements in methods and techniques (see Blottner[4]) for solving the boundary-layer equations should therefore be of use in solving the Navier-Stokes equations.

The second development was that of Van Dyke[5], who found that there are similarity type variables for some problems involving the Navier-Stokes equations. He developed a method of series truncation which assumes that when the problem is written in the proper coordinate system, variables almost separate, and approximate solutions can be obtained by neglecting the variation with respect to the coordinate with weak dependence. This results in a set of ordinary differential equations which can be solved numerically. Davis[6] has used a modification of this method to solve for laminar incompressible flow past a semi-infinite flat plate. In choosing the proper coordinate system the work of Kaplun[7] on optimal coordinates is undoubtedly of great importance.

The third development is due to Cheng,[8] Davis,[9] Cheng, Chen, Mobley, and Huber,[10] and others, who found that the full Navier-Stokes equations can, in some cases, be approximated by a set of parabolic equations which can be solved using the same techniques as are used in solving the boundary-layer equations. They were concerned with solving problems involving various forms of the hypersonic viscous shock-layer equations; however parabolic type equations may arise as approximations in other viscous flow problems as well. It would therefore be useful to start with these approximate solutions and systematically relax them to obtain an exact solution to the full Navier-Stokes equations.

When one wishes to take advantage of all of the developments mentioned above, one searches for a numerical technique in which they can be used. The obvious one is a modification of the implicit alternating direction method due to Douglas,[11] since the integration in the spatial variables can be made almost identical to the implicit method for solving the boundary-layer equations. This new technique has been applied with success by Davis[12] to the problem of laminar incompressible flow past a parabola.

In this paper, we demonstrate the new method by solving the problem of laminar flow past a paraboloid of revolution. This problem is picked since it can be used to demonstrate each aspect of the present method.

## Governing Equations and Boundary Conditions

As in the plane case of flow past a parabola (see Davis[12]), parabolic coordinates seem to be the best for flow past a paraboloid of revolution. In these coordinates variables separate for the low Reynolds number case to provide the Oseen solution in closed form. Wilkinson[13] has found the Oseen solution for the elliptic paraboloid for which the present problem is a special case. Parabolic coordinates also provide equations for which the variables separate to provide the asymptotic solution to the full Navier-Stokes equations far downstream on the paraboloid of revolution. This problem has been studied by Mark,[14] Mather,[15] Albacete,[16] Lee,[17] Cebeci, Na and Mosinski,[18] and Miller.[19]

The parabolic coordinates are defined by the relationship

$$x + iy = \nu(\xi + i\eta)^2/2U \tag{1}$$

where the $x, y$ coordinates are dimensional Cartesian coordinates and the $\xi, \eta$ coordinates are dimensionless parabolic coordinates. The quantity $\nu$ is the kinematic viscosity and $U$ is the freestream velocity.

All parameters can be removed from the governing equations by nondimensionalizing the time t by $\nu/U^2$, the stream function $\psi$ by $\nu^2/U$ and the vorticity $\omega$ by $U^2/\nu$. In these variables the paraboloid surface is located at $\eta = Re^{1/2}$ where $Re = Ua/\nu$, $a$

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being the paraboloid nose radius of curvature. With these substitutions the governing equations in terms of stream function and vorticity become:

$$\left[(\xi^2+\eta^2)D^2+\frac{1}{\xi\eta}\left(\psi_\xi\frac{\partial}{\partial\eta}-\psi_\eta\frac{\partial}{\partial\xi}\right)+\frac{2}{\xi^2\eta^2}(\eta\psi_\eta-\xi\psi_\xi)-(\xi^2+\eta^2)\frac{\partial}{\partial t}\right]D^2\psi=0 \quad (2)$$

and

$$D^2\psi+\xi\eta\omega=0 \quad (3)$$

where

$$D^2=\frac{\xi\eta}{\xi^2+\eta^2}\left[\frac{\partial}{\partial\xi}\left(\frac{1}{\xi\eta}\frac{\partial}{\partial\xi}\right)+\frac{\partial}{\partial\eta}\left(\frac{1}{\xi\eta}\frac{\partial}{\partial\eta}\right)\right] \quad (4)$$

with boundary conditions

$$\psi\sim\xi^2\eta^2/2 \text{ as } \eta\to\infty \quad (5)$$

and

$$\left.\begin{aligned}\psi&=0 \quad (6)\\ \psi_\eta&=0 \quad (7)\end{aligned}\right\}\text{at } \eta=Re^{1/2}$$

With these definitions the dimensionless velocity components (nondimensionalized by $U$) in the $\xi$ and $\eta$ coordinate directions respectively become

$$u=1/[\xi\eta(\xi^2+\eta^2)^{1/2}]\,\partial\psi/\partial\eta \quad (8)$$

and

$$v=-1/[\xi\eta(\xi^2+\eta^2)^{1/2}]\,\partial\psi/\partial\xi \quad (9)$$

In order to find the surface pressure we evaluate the $\xi$ momentum equation on the body surface. Defining the pressure to be the pressure minus $P_\infty$ and nondimensionalizing by $\rho U^2$ we find on the body surface

$$\frac{\partial P}{\partial\xi}=\frac{1}{(\xi^2+\eta^2)^{1/2}}\frac{\partial^2u}{\partial\eta^2}+\frac{1}{\eta(\xi^2+\eta^2)^{1/2}}\frac{\partial u}{\partial\eta}\text{ at } \eta=Re^{1/2} \quad (10)$$

It can be shown that $P$ at $\xi$ equal to infinity is zero since the asymptotic solution far downstream will show that pressure is constant with respect to $\eta$. Thus, once the solution for $\psi$ is found the body surface pressure can be calculated with Eqs. (8) and (10) by integrating from downstream infinity back along the body surface.

We now introduce new dependent variables that almost separate to produce ordinary differential equations. These are a set of appropriate variables for finding locally similar solutions (see Davis[6] and Davis[12] for the equivalent variables for the parabola problem). This substitution is the appropriate one to produce the low Reynolds number Oseen solution of Wilkinson[13] and also the asymptotic solution to the full Navier-Stokes equations far downstream. They would also be the appropriate variables to use for a series truncation solution using Van Dyke's[5] approach. Thus, using these variables will result in a slow variation with respect to the $\xi$ direction and result in better accuracy in the numerical method. They will also produce an approximate starting solution at the nose of the paraboloid. Thus, we introduce

$$\psi=\xi^2f(\xi,\eta) \quad (11)$$

and

$$\omega=-\xi/[\eta(\xi^2+\eta^2)]\,g(\xi,\eta) \quad (12)$$

We are led to the first substitution Eq. (11) from the boundary conditions (5). The second substitution Eq. (12), results from looking at Eq. (3) after the substitution of Eq. (11) has been made.

The new set of governing equations and boundary conditions are

$$\begin{aligned}g_{\eta\eta}&+[(1/\eta)(2f+\xi f_\xi)-(\xi^2+5\eta^2)/\eta(\xi^2+\eta^2)]g_\eta\\&+[2/\eta^2(\xi^2+\eta^2)][\xi^2\eta f_\eta-(\xi^2+2\eta^2)(2f+\xi f_\xi)]g\\&-[(\xi/\eta)f_\eta+4\xi/(\xi^2+\eta^2)]g_\xi-(\xi^2+\eta^2)g_t\\&+g_{\xi\xi}+(3/\xi)g_\xi=0 \quad (13)\end{aligned}$$

and

$$f_{\eta\eta}-(1/\eta)f_\eta-g+f_{\xi\xi}+(3/\xi)f_\xi=0 \quad (14)$$

with boundary conditions

$$\left.\begin{aligned}f&\sim\eta^2/2 \quad (15)\\ g&\sim 0 \quad (16)\end{aligned}\right\}\text{as } \eta\to\infty$$

and

$$\left.\begin{aligned}f&=0 \quad (17)\\ f_\eta&=0 \quad (18)\end{aligned}\right\}\text{at } \eta=Re^{1/2}$$

The resulting velocity and pressure expressions become from Eqs. (8–10)

$$u=[\xi/\eta(\xi^2+\eta^2)^{1/2}]f_\eta \quad (19)$$

$$v=[-1/\eta(\xi^2+\eta^2)^{1/2}](2f+\xi f_\xi) \quad (20)$$

and

$$\frac{\partial P}{\partial\xi}=\frac{\xi}{\eta(\xi^2+\eta^2)}\left[g_\eta-\frac{2\eta}{\xi^2+\eta^2}g\right]\text{ at } \eta=Re^{1/2} \quad (21)$$

Ignoring the derivatives with respect to $\xi$ in Eqs. (13) and (14) we obtain after some algebra

$$\eta\left[\frac{1}{\eta^2}(\eta g'+2fg)\right]'-\frac{4}{\xi^2+\eta^2}\left[\eta g'+\left(f+\frac{\eta f'}{2}\right)g\right]=0 \quad (22)$$

and

$$g=\eta(f'/\eta)' \quad (23)$$

where primes denote differentiation with respect to $\eta$.

Because of the $\xi$ term in the denominator of the coefficient of the second term in Eq. (22), variables do not separate. However, in two special limiting cases they do. We will discuss these limiting cases in the following.

Letting $\xi$ go to infinity, the second term in Eq. (22) drops out to produce an ordinary differential equation. The solution to this equation along with Eq. (23) gives the asymptotic solution far downstream. Integrating the resulting equation once and using the boundary conditions of Eqs. (15) and (16) gives

$$\eta g'+2fg=0 \quad (24)$$

and

$$g=\eta[f'/\eta]' \quad (25)$$

with boundary conditions

$$f=f_\eta=0 \text{ at } \eta=Re^{1/2} \quad (26)$$

and

$$f_\eta\sim\eta \text{ as } \eta\to\infty \quad (27)$$

The solutions of these equations provide the proper downstream boundary conditions for solution of the full Navier-Stokes equations. Note that as Reynolds number goes to infinity they reduce to the Blasius equation as they should. The extra terms which appear at finite Reynolds number represent transverse curvature effects.

An asymptotic solution can also be found to Eqs. (24–27) for small Reynolds number. The method of matched asymptotic expansions (see Van Dyke[5]) has been used since a Stokes type expansion governs the flow near the body, whereas the outer flow is governed by an Oseen type expansion. Three terms of both the inner and outer expansion have been calculated and matched to determine the skin-friction function $g$ at the body surface. Van Dyke[20] has previously found the two term solution and our result is in complete agreement with his. The method of approach used here is exactly the same as that of Van Dyke, except that we have separated the variables first and then perturbed the governing Eq. (24) rather than perturb the governing equations and then separate variables. We know that the solution we obtain using Eq. (24) will be valid far downstream. We must examine the result to determine to what extent the solution is valid for arbitrary $\xi$.

The three term Stokes expansion for the skin-friction function gives at the body surface

$$g(\infty, Re^{1/2}) = 2\varepsilon[1 - (ln2 - \gamma)\varepsilon - (2ln2 + \pi^2/12) - \{ln2 - \gamma\}^2)\varepsilon^2 + \ldots] \quad (28)$$

where

$$\varepsilon = 1/ln\,(1/\mathrm{Re}) \quad (29)$$

and $\gamma$ is Euler's constant whose value is 0.5772157.

Mark[14] has given an expression similar to Eq. (28) except in terms of another small parameter $\tau$ which can be written as

$$\tau = \varepsilon/[1 + (ln2 - \gamma)\varepsilon \quad (30)$$

Rewriting Eq. (28) in terms of $\tau$ gives

$$g(\infty, Re^{1/2}) = 2\tau[1 - (2ln2 + \pi^2/12)\tau^2 + \ldots] \quad (31)$$

which is exactly the same as Mark's result.

In order for Eq. (28) to be valid for all $\xi$ at low Reynolds number, we must show that it is an asymptotic solution to the second term in brackets in Eq. (22), i.e.

$$\eta g' + (f + \eta f'/2)g = 0 \quad (32)$$

The same type of matched asymptotic solution has been found to this equation along with Eqs. (24–27). To the second approximation the results are identical in both the inner and outer expansions. In the third approximation they differ indicating that variables do not separate at the third-order level. The solution to Eq. (32) rather than Eq. (24) results in $\pi^2/12$ being replaced by $\pi^2/24$ in Eq. (28). All other quantities in Eq. (28) are the same. This is not a large effect, but to determine the actual third-order contribution one must return to Eqs. (13–18) to find it. It will be shown later that even though Eq. (28) may not be a uniformly valid solution to third-order for all $\xi$, it shows good agreement with the numerical solution for all positions on the paraboloid surface.

Equations (22) and (23) can also be used to produce the Oseen solution. At low Reynolds number the Oseen approximation involves replacing the $f$ in Eq. (22) by $\eta^2/2$ from Eq. (15). The terms in brackets in Eq. (22) thus become the same, and due to the boundary conditions given in Eqs. (15) and (16), variables again separate to give

$$g' + \eta g = 0 \quad (33)$$

and

$$g = \eta[f'/\eta]' \quad (34)$$

Equation (33) can be integrated to give

$$g = Ce^{-\eta^2/2} \quad (35)$$

where $C$ is a constant of integration.

Substituting Eq. (35) into Eq. (34), integrating once, and using the boundary condition, Eq. (18), we obtain

$$f' = C\eta \int_{\mathrm{Re}^{1/2}}^{\eta} (e/\zeta)^{-(\zeta^2/2)}\, d\xi \quad (36)$$

Using the boundary conditions of Eqs. (15) we evaluate $C$ to obtain

$$C^{-1} = \int_{\mathrm{Re}^{1/2}}^{\infty} e^{-(\zeta^2/2)}/\zeta d\zeta \quad (37)$$

which can be written in terms of the exponential integral as

$$C^{-1} = E_1(Re/2)/2 \quad (38)$$

Thus, Eq. (36) can be written as

$$f' = \eta[E_1(Re/2) - E_1(\eta^2/2)]/E_1(Re/2) \quad (39)$$

The function $g$ in Eq. (35) evaluated at the wall, which is related to the wall skin friction, is given by

$$g(\xi, Re^{1/2}) = 2e^{-\mathrm{Re}/2}/E_1(Re/2) \quad (40)$$

where the skin-friction coefficient $C_f$ is given by

$$C_f = \tau/\rho U^2 = \xi/[Re^{1/2}(\xi^2 + Re)]\, g(\xi, Re^{1/2}) \quad (41)$$

Equation (40) is the Oseen result which has been given by Wilkinson.

If Eq. (40) is expanded for small Reynolds number it can be shown that it gives the same result to second order as Eq. (28). Thus, the Oseen solution is a uniformly valid second-order solution. However it will be shown in the numerical results later that its range of applicability is limited to extremely small Reynolds numbers, making it, for all practical purposes, useless. The situation is even more drastic in the plane case where Davis[12] has shown that the Oseen solution fails in the limit of zero Reynolds number flow past a parabola.

For the low Reynolds number case we can integrate Eq. (21) to find the surface pressure. Evaluating Eq. (24) at the body surface we find $g' = 0$. Using this in Eq. (21) we find

$$\partial P/\partial \xi = -[2\xi/(\xi^2 + \eta^2)^2]g \text{ at } \eta = Re^{1/2} \quad (42)$$

From Eq. (28) or (40), we have shown that $g$ is not a function of $\xi$ at low Reynolds numbers. Thus

$$P(\xi, Re^{1/2}) = \frac{g(\xi, Re^{1/2})}{\xi^2 + \eta^2} \quad (43)$$

at low Reynolds number. Note that we have used the boundary condition that $P \to 0$ as $\xi \to \infty$.

These results are given to demonstrate that the governing Eqs. (13–18) automatically produce the correct asymptotic results. The numerical method of solution will be such that it will approach the correct limiting values. The analytic solutions can thus be used to determine the accuracy of the numerical method.

## Numerical Method of Solution

In this section we discuss the numerical method of solution of Eqs. (13–18). The method has been previously discussed briefly by Davis.[12] Here we present additional details of the method.

Because of the boundary condition given in Eq. (15) we replace $f$ by

$$h = f - \eta^2/2 \quad (44)$$

This avoids the difficulty of having $f$ go to infinity as $\eta$ goes to infinity. The function $h$ is found to be bounded in the whole region between the body and the freestream, and is thus convenient for use in the numerical method.

As in the parabola case, studied by Davis,[12] it has been found that dropping the $(\xi^2 + \eta^2)$ coefficient from the time derivative in Eq. (13) results in faster convergence in the time dependent numerical method. Since we are only interested in the steady-state solution, this can be done.

The solution in the alternating direction implicit method is achieved in two steps. First we assume the starred (*) terms on the left-hand side of the following equations to be unknown and the terms with the subscripts $j$ on the right-hand side to be known. The governing equations and boundary conditions for this half time step are written as follows:

$$g^*_{\eta\eta} + \left[\frac{1}{\eta}(2h^* + \eta^2 + \xi h^*_\xi) - \frac{(\xi^2 + 5\eta^2)}{\eta(\xi^2 + \eta^2)}\right] g^*_\eta + \left[\frac{2}{\eta^2(\xi^2 + \eta^2)}\left[\xi^2(\eta h^*_\eta + \eta^2) - (\xi^2 + 2\eta^2)(2h^* + \eta^2 + \xi h^*_\xi)\right] - \frac{2}{\Delta t}\right] g^* - \left[\frac{\xi}{\eta}(h^*_\eta + \eta) + \frac{4\xi}{\xi^2 + \eta^2}\right] g^*_\xi = -\frac{2}{\Delta t} g_j - \left[g_{\xi\xi} + \frac{3}{\xi} g_\xi\right]_j \quad (45)$$

and

$$h^*_{\eta\eta} - (1/\eta)h^*_\eta - g^* - (2/\Delta v)h^* = -2h_j/\Delta v - [h_{\xi\xi} + (3/\xi)h_\xi]_j \quad (46)$$

with

$$h(\xi, Re^{1/2}) = -Re/2 \quad (47)$$

$$h_\eta(\xi, Re^{1/2}) = -Re^{1/2} \quad (48)$$

$$h_\eta(\xi, \infty) = 0 \quad (49)$$

and

$$g(\xi, \infty) = 0 \tag{50}$$

Note that a fictitious unsteady term of $-\partial h/\partial v$ has been added to Eq. (46) in order to use the alternating direction method.

The Eqs. (45) and (46) have been deliberately written so that the left-hand sides form a set of parabolic partial differential equations in the spatial coordinates $\xi, \eta$. These left-hand sides include all of the terms which would appear in second-order boundary-layer theory with the exception of displacement effects. They also include the low Reynolds number Oseen solution and the correct asymptotic solution far downstream, and neglecting the $\xi$ derivatives produces the locally similar results. Therefore, the solution to the problem should show very strong dependence on the left-hand sides and weak dependence on the right-hand sides. In fact, as a starting solution at time equals zero the right-hand sides are completely neglected along with the $2g^*/\Delta t$ and $2h^*/v$ terms on the left-hand sides of the equations. The resulting parabolic partial differential equations are then integrated starting at the stagnation point to downstream infinity using the implicit finite difference method of Blottner and Flügge-Lotz[1] for solving the boundary-layer equations. Any other finite difference method for solving boundary-layer equations would also be appropriate at this stage and thus more refined boundary-layer numerical methods would be advantageous. We use the simplest technique here only to demonstrate the method. It is important to note that this step produces the correct asymptotic solution far downstream which will be held as a downstream boundary condition as the computation proceeds from this initial solution.

In subsequent half-time steps involving the star (*) step the right hand sides of Eqs. (45) and (46) are not set equal to zero the values from the $i$ step are put in. Care must be taken at $\xi$ equals zero with the $(3g_\xi/\xi)_j$ and $(3h_\xi/\xi)_j$ terms and they are replaced by their limiting values of $(3g_{\xi\xi})_j$ and $(3h_{\xi\xi})_j$ respectively. In addition, the $-2g^*/\Delta t$ and $-2h^*/\Delta v$ terms are not set equal to zero. Otherwise the method of attack is exactly the same, using the implicit boundary-layer method to solve the equations.

Because the boundary conditions (47–49) do not include a relation for $g(\xi, Re^{1/2})$, the Thomas[21] algorithm can not be used directly to solve the resulting difference equation. This is overcome by superimposing a homogeneous and particular solution so that Eq. (48) is satisfied, thus determining $g(\xi, Re^{1/2})$.

At the next half time step the governing Eqs. (13) and (14) are written as:

$$\begin{aligned}
[g_{\xi\xi} + (3/\xi)g_\xi]_{j+1} - 2g_{j+1}/\Delta t = &-g^*_{\eta\eta} \\
&- [(1/\eta)(2h^* + \eta^2 + \xi h^*_\xi) - (\xi^2 + 5\eta^2)/\eta(\xi^2 + \eta^2)]g^*_\eta \\
&- \{[2/\eta^2(\xi^2 + \eta^2)][\xi^2(\eta h^*_\eta + \eta^2) \\
&- (\xi^2 + 2\eta^2)(2h^* + \eta^2 + \xi h^*_\xi)] \\
&+ 2/\Delta t\}g^* + [(\xi/\eta)(h^*_\eta + \eta) + 4\xi/(\xi^2 + \eta^2)]g^*_\xi
\end{aligned} \tag{51}$$

and

$$\begin{aligned}
[h_{\xi\xi} + (3/\xi)h]_{j+1} - 2h_{j+1}/\Delta v \\
= -h^*_{\eta\eta} + (1/\eta)h^*_\eta + g^* - 2h^*/\Delta v
\end{aligned} \tag{52}$$

The left sides of Eqs. (51) and (52) are now written as three point central differences and the right-hand sides are known from the previous star step of the alternating direction method. The equations can thus again be solved using the Thomas[21] algorithm if boundary conditions at $\xi$ equals zero and infinity are given. The downstream conditions at $\xi$ equals infinity are obtained from the initial starting solution, as mentioned previously, since the numerical method automatically produces the correct asymptotic result as was shown by Eqs. (24) and (25). The conditions at $\xi$ equals zero are obtained by evaluating Eqs. (51) and (52) at $\xi$ equals zero, recognizing the limiting values of $(3g_\xi/\xi)_{j+1}$ and $(3h_\xi/\xi)_{j+1}$ as $(3g_{\xi\xi})_{j+1}$ and $(3h_{\xi\xi})_{j+1}$, respectively. The resulting equations produce boundary conditions by writing $(g_{\xi\xi})_{j+1}$ and $(h_{\xi\xi})_{j+1}$ as three point central differences and recognizing the symmetry of the $g_{j+1}$ and $h_{j+1}$ functions about the stagnation point. The resulting difference equations are than appropriate for use with the Thomas[21] algerithm.

The steady-state solution to the problem is then found by alternating between the star and $j+1$ steps of the method.

In the present method, steps are taken simultaneously in $\Delta t$ and $\Delta v$ and thus each step does not represent a true unsteady solution. This is done in order to get to the steady-state solution with as little computing time as possible. However, the method could be made a truly unsteady method by restoring the $(\xi^2 + \eta^2)$ coefficient in the $g_t$ term and then relaxing the solution with steps in $\Delta v$ for each real time step $\Delta t$.

As in the parabola case investigated by Davis,[12] it is convenient to redefine the independent variables so that the boundary conditions can be applied at infinity rather than at finite values of $\xi$ and $\eta$. This is necessary since the stream function $h$ dies out to its freestream value algebraically. Since the vorticity function $g$ dies out exponentially, it is not necessary to apply the boundary conditions on it at infinity; however, for convenience in the present problem we do.

The same transformations on the dependent variables as were used by Davis[12] in the parabola problem are used here. Similar transformations have been used by van de Vooren and Dijkstra[22] and Veldman and Dijkstra.[23] Transformations of the same type have been shown by van de Vooren and Dijkstra[22] to be free of singularities at infinity in the parabola problem.

The new variables $S$, $N$ are defined by:

$$\eta = 5N/(1 - N) + Re^{1/2} \tag{53}$$

and

$$S = 1 - \log(1 + \xi/A)/(\xi/A) \tag{54}$$

where

$$A = 4 + 0.4Re^{1/2} \tag{55}$$

The new $S$, $N$ variables run from 0,0 to 1,1 whereas $\xi,\eta$ run from 0, $Re^{1/2}$ to $\infty,\infty$. The 5 appears in Eq. (53) so that the edge of the viscous region (the point where $g$ goes to zero) is located at $N$ approximately equals 0.5. The quantity $A$ in Eq. (55) has been defined such that $S$ equals 0.5 out approximately half way between the value of $g$ at the stagnation point and downstream infinity.

With these definitions the governing equations are integrated numerically. Details of the convergence of the method and presentation of results will be given in the next section.

## Results and Conclusions

The governing equations for flow past a paraboloid of revolution have been integrated using the method described above. In the first half time step Eqs. (45–50) are used followed in the second half time step by Eqs. (51–52). All calculations were made with 125 steps in the $N$ direction and 20 steps in the $S$ direction. These seemed to be a sufficient number of steps to produce three place accuracy.

We are only interested in the final steady-state solution and not the transient solution. For this reason we are interested in getting to the steady-state solution with as little computing time as possible. To determine how this can be done we have run numerical experiments with the two time steps $\Delta t$ and $\Delta v$ the same to determine the rate of convergence as a function at $\Delta t$. We have chosen an intermediate value of Reynolds number of ten and run the program for various values of $\Delta t$. To monitor the convergence we have chosen the stagnation point value of $g$, i.e. $g$ $(0, Re^{1/2})$, from the star step of the alternating direction method. Some preliminary calculations were made to approximately determine the most efficient $\Delta t$ step to obtain fast convergence.

Figure 1 indicates the rate of convergence using several values of $\Delta t$. Note that the maximum rate of convergence occurs somewhere around a value of $\Delta t$ of five. Convergence is quite fast for this time step and three place accuracy is achieved in about ten iterations. In the actual computations, the values of $g$ are checked at all points in the flowfield and the computation is terminated when the solution is no longer varying, to some prescribed accuracy, with further iterations.

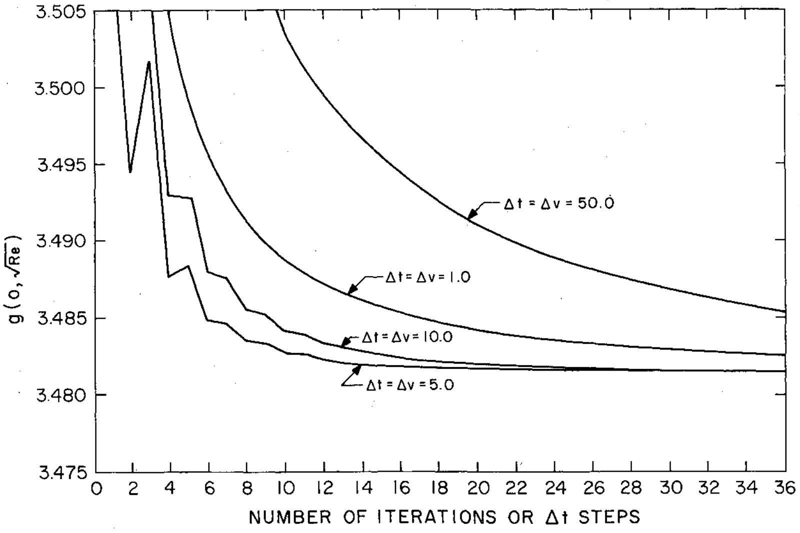


**Fig. 1 Convergence of the method at $Re = 10.0$.**

The computing time using double precision arithmetic on the IBM 360-65 was less than one minute per time step, meaning that convergence can be achieved in less than ten minutes computing time.

The optimal time step $\Delta t$ is a function of Reynolds number; however, a value of ten was used for all subsequent calculations regardless of Reynolds number. Very fast convergence is achieved at high Reynolds numbers, which should be obvious from the way the problem is formulated to include the boundary-layer equations in the star step of the alternating direction method. Low Reynolds number calculations also converged rapidly since dependence on $\xi$ drops out of the equations as Reynolds number goes to zero. The intermediate Reynolds number cases tended to show slowest convergence, which is why the optimal time step was determined at a Reynolds number of ten.

No attempt was made to further optimize the method by taking values of $\Delta v$ not equal to $\Delta t$. This could possibly lead to even faster convergence.

The rapid convergence in the present problem is due to several factors which were mentioned in the introduction. The use of similarity type variables along with a boundary-layer type alternating direction method are the primary factors. The method would probably not be as useful in many other problems where similarity type variables cannot be found. However, just as one cannot rely on one analytical technique to solve all problems, the same is true of numerical techniques. The present method should therefore be considered in that way and its usefulness in a broad class of problems should be determined before any definite statements can be made.

Figure 2 shows the skin-friction function evaluated at the leading edge and also at downstream infinity as a function of Reynolds number. Comparison is made with one, two, and three terms of the Stokes solution given by Eq. (28). This asymptotic solution is only valid to two terms for arbitrary $\xi$. However, Fig. 2 indicates good agreement between the present results and the third-order asymptotic solution for all $\xi$ as Reynolds number goes to zero. This is probably due to the fact that there should be only a small difference between the solution for arbitrary $\xi$ and the solution at $\xi$ equals infinity.

By far the best approximation shown is the local similarity solution given by Davis.[24] It approaches the exact solution as $\xi$ goes to infinity, as Reynolds number goes to infinity at $\xi$ equals zero. This solution is obtained by integrating Eqs. (22) and (25–27) for the proper value of $\xi$ and Reynolds number. The solution at $\xi$ equals infinity is in agreement with the solutions obtained by Cebeci, Na and Mosinskis[18] and others.

It should be noted from Fig. 2 that while the Oseen solution approaches the correct solution as Reynolds number goes to zero, at a Reynolds number of 0.01 it is in error by about 10%. For higher Reynolds numbers the error increases rapidly making the Oseen solution useless for any practical application.

Figure 3 shows the stagnation point pressure as a function of Reynolds number as obtained from Eq. (21). In performing the numerical integration the singularity encountered as Reynolds number goes to zero, indicated by Eq. (42), has been removed to obtain better accuracy. As indicated in the figure the Oseen solution gives a rather poor result.

Figure 4a shows the skin-friction function for various Reynolds numbers, compared with the local similarity results of Davis.[24] Local similarity is exact as $\xi$ goes to infinity as the figures indicate. Figure 4b shows a comparison with the "exact" boundary-layer results of Davis[25] which were obtained using a finite-

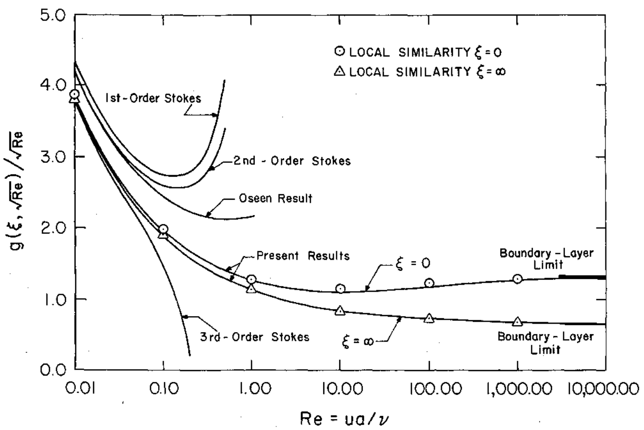


**Fig. 2 Skin friction at the stagnation point and far downstream on a paraboloid of revolution.**

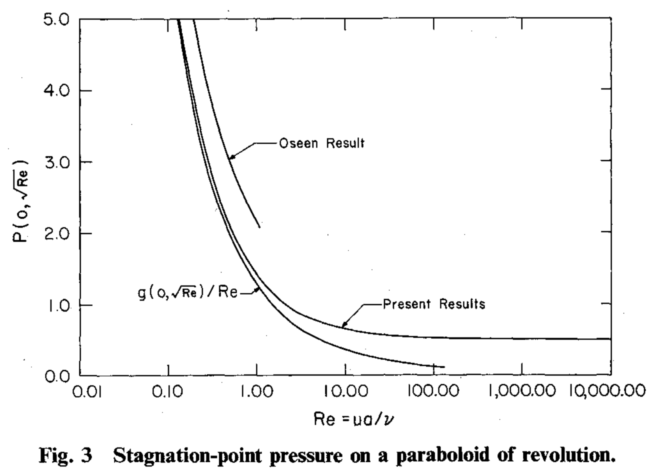


**Fig. 3 Stagnation-point pressure on a paraboloid of revolution.**

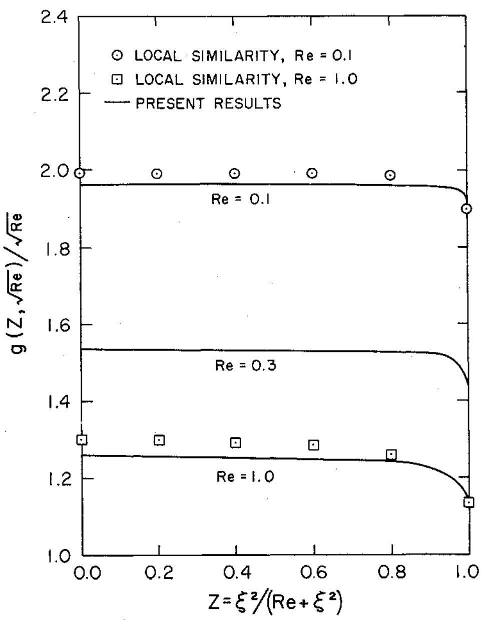


**Fig. 4a Skin-friction distribution on a paraboloid of revolution.**

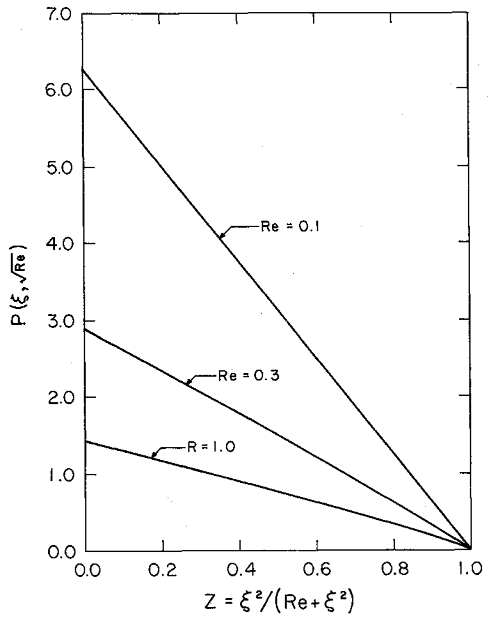


**Fig. 5a Pressure distribution on a paraboloid of revolution.**

difference method. The present results appear to be approaching the boundary-layer limit.

Figure 5a shows the pressure distributions on the paraboloid. One should note in Fig. 5b that the correct inviscid limit is approached and is indicated by $P_i$. At a Reynolds number of 1,000 a slight difference is shown between the inviscid pressure $P_i$, and the present results. This is consistent with what one expects from second-order, boundary-layer theory where it is found that there is no second-order correction at the stagnation point, but there is one downstream.

In conclusion, a new numerical technique has been developed for solving the Navier-Stokes equations. The method converges rapidly and produces results for flow past a paraboloid of revolution which approach all of the correct limiting values. Before conclusions can be drawn about the applicability of the method as a general technique, other more difficult problems must be solved using the method.

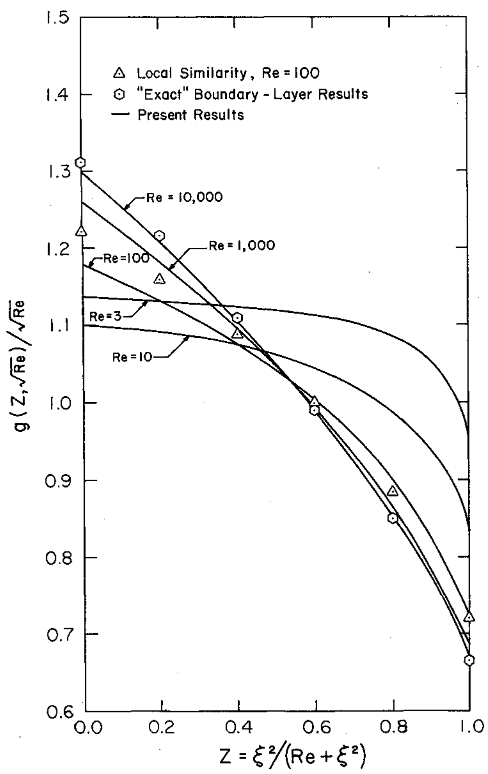


**Fig. 4b Skin-friction distribution on a paraboloid of revolution.**

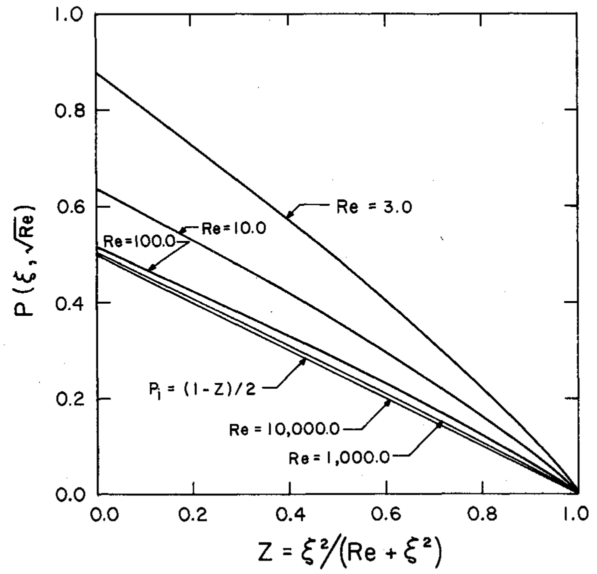


**Fig. 5b Pressure distribution on a paraboloid of revolution.**

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# Nonsimilar Solution for Laminar and Turbulent Boundary-Layer Flows over Ablating Surfaces

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**A mathematical model and numerical solution of nonsimilar laminar or turbulent multicomponent chemically reacting boundary-layer flows are presented. The general flow model and solution technique represent an extension of a previously presented laminar flow scheme to the turbulent regime and to a broader set of boundary conditions. The turbulent model, which includes a mixing length representation in the wall region and a constant eddy viscosity in the wake region, represents a compressible flow adaption of a previously validated incompressible turbulent model. Boundary conditions for the solution procedure include features which are most useful for re-entry calculations with ablation, such as direct coupling with an entropy layer, with an ablating wall or with surface injection. Sample problems have been selected which illustrate these features, demonstrate the necessity for a nonsimilar analysis, and validate the compressible flow turbulent model.**

## Nomenclature

$c_t$ = thermal diffusion constant (Ref. 4)
$C$ = density-viscosity product normalized by reference values
$\bar{C}_p$ = frozen specific heat of the gas mixture
$\tilde{C}_p$ = gas mixture property which reduces to $\bar{C}_p$ when all diffusion coefficients are equal (Ref. 4)
$C_{p_i}$ = specific heat of species $i$
$\bar{D}$ = reference binary diffsion coefficient of Ref. 6
$D_i^T$ = multicomponent thermal diffusion coefficient for species $i$
$D_{ij}$ = multicomponent diffsion coefficient for species $i$ and $j$
$\mathscr{D}$ = diffusion coefficient for all species when all $\mathscr{D}_{ij}$ are equal
$\mathscr{D}_{ij}$ = binary diffusion coefficient for species $i$ and $j$
$f$ = stream function
$h$ = static enthalpy of the gas
$\tilde{h}$ = gas mixture property which reduces to the static enthalpy $h$ when all diffusion coefficients are equal
$h^o$ = heat of formation
$H_T$ = total enthalpy
$j_k$ = diffusional mass flux of element $k$ per unit area away from the surface

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